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WELL-POISED HYPERGEOMETRIC TRANSFORMATIONS OF EULER-TYPE MULTIPLE INTEGRALS

W. ZUDILIN

Abstract

Several new multiple-integral representations are proved for well-poised hypergeometric series and integrals. The results yield, in particular, transformations of the multiple integrals that cannot be achieved by evident changes of variable. All this generalizes some classical results of Whipple and Bailey in analysis and, on the other hand, certain analytic constructions with known connection to irrationality proofs for values of Riemann's zeta function at positive integers.

1. *Motivation from number theory*

In 1978, Apéry showed [**2**] that the values of $\zeta(s)=\sum_{n=1}^\infty n^{-s}$ at $s=2$ and $s=3$ are irrational. Although the irrationality of $\zeta(2)=\pi^2/6$ was known, Apéry's novel method for proving irrationality ('accélération de la convergence', cf. [**8**]) gave a new proof for $\zeta(2)$.

One year later, Beukers [**7**] suggested the Euler-type multiple integrals

$$J_{2,n}=\iint\limits_{[0,1]^2}\frac{x^n(1-x)^ny^n(1-y)^n}{(1-xy)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

and

$$J_{3,n}=\iiint\limits_{[0,1]^3}\frac{x^n(1-x)^ny^n(1-y)^nz^n(1-z)^n}{\bigl(1-(1-xy)z\bigr)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2}$$

for Apéry's approximations [**2**] to $\zeta(2)$ and $\zeta(3)$, respectively. Beukers proved that

$$D_n^2J_{2,n}\in\mathbb{Z}\zeta(2)+\mathbb{Z}\qquad\text{and}\qquad D_n^3J_{3,n}\in\mathbb{Z}\zeta(3)+\mathbb{Z} \tag{3}$$

where D_n denotes the least common multiple of the numbers $1,2,\dots,n$ and that the positive quantities $D_n^2J_{2,n}$, $D_n^3J_{3,n}$ tend to 0 as $n\to\infty$. The simplicity of Beukers's proof [**7**] of Apéry's theorem led to further improvements of estimates for the irrationality measures of $\zeta(2)$ and $\zeta(3)$ by means of similar multiple integrals; Rhin and Viola are the last record-holders in this direction (see [**14**], [**15**] and [**22**]). Consideration of the general family of integrals

$$J_{k,n}=\int\cdots\int\limits_{[0,1]^k}\frac{\prod_{j=1}^k x_j^n(1-x_j)^n}{Q_k(x_1,\dots,x_k)^{n+1}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \tag{4}$$

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with

$$Q_k(x_1,\dots,x_k)=1-x_1(1-x_2(1-\dots(1-x_{k-1}(1-x_k))\cdots)) \tag{5}$$

was proposed by Vasilenko [**20**]. Recently, Vasilyev studied [**21**] the integrals $J_{4,n}$ and $J_{5,n}$ and proved that

$$4D_n^4J_{4,n}\in\mathbb{Z}\zeta(4)+\mathbb{Z}\zeta(2)+\mathbb{Z},\qquad D_n^5J_{5,n}\in\mathbb{Z}\zeta(5)+\mathbb{Z}\zeta(3)+\mathbb{Z}, \tag{6}$$

and that the linear forms in (6) tend (rather fast) to 0 as $n\to\infty$ (unfortunately, not sufficiently rapidly to obtain new irrationality results).

On the other hand, Gutnik [**10**] and later Nesterenko [**12**] suggested another representation of Apéry's approximants, in terms of hypergeometric functions and so-called Meijer's G-functions (Barnes-type hypergeometric integrals)

$$G_{k,k}^{m,n}\left(\begin{matrix}a_1,\dots,a_k\\ b_1,\dots,b_k\end{matrix}\,\middle|\,z\right)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^n\Gamma(1-a_j+t)\prod_{j=1}^m\Gamma(b_j-t)}{\prod_{j=n+1}^k\Gamma(a_j-t)\prod_{j=m+1}^k\Gamma(1-b_j+t)}\,z^t\,\mathrm{d}t, \tag{7}$$

where the path of integration is parallel to the imaginary axis, except that it is curved, if necessary, so that the decreasing sequences of poles of the functions $\Gamma(1-a_j+t)$, $j=1,2,\dots,n$, lie to the left of the contour, while the increasing sequences of poles of the functions $\Gamma(b_j-t)$, $j=1,2,\dots,m$, lie to the right. All complex contours used in this paper are of this type; such integrals were introduced by Barnes [**6**] in order to deduce analytic continuation, summation and transformation formulae for generalized hypergeometric series. The conditions $m+n-k>0$ and $\sum_{j=1}^k\operatorname{Re}(b_j-a_j)<-1$ ensure the absolute convergence of the integral (7) in the domain $|\arg z|\le(m+n-k)\pi$ (see [**11**], Section 5.3.1).

Recently, Nesterenko proved a general theorem [**13**] yielding representation of the G-functions (7) in terms of Euler-type integrals. In particular, the theorem in [**13**] makes a bridge between Beukers's and Gutnik's proofs of Apéry's theorem. Another recent contribution, due to Ball (see the introduction in [**16**]), is to introduce the (very special type of) hypergeometric series

$$F_{3,n}=n!^2\sum_{t=1}^\infty(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{t^4(t+1)^4\cdots(t+n)^4} \tag{8}$$

satisfying the easily verified property $F_{3,n}\in\mathbb{Q}\zeta(3)+\mathbb{Q}$. The mysterious coincidence $F_{3,n}=J_{3,n}$ was shown by Rivoal [**17**] by application of the human telescoping algorithm of Zeilberger. Generalizing the series in (8) and the proof in [**12**] of Apéry's theorem, Rivoal was able to prove [**16**] that infinitely many of the values $\zeta(s)$ at $s=3,5,7,\dots$ are irrational.

In [**23**], Whipple gave the name *well-poised* to the class of hypergeometric series having parameters in pairs with the same sum. It is a historical fact that well-poised hypergeometric series and integrals play a special role in the subject (see [**1**]). The series in (8) is well-poised, and the equality $F_{3,n}=J_{3,n}$ can be also deduced by rather classical means—Bailey's transformation [**4**] and Nesterenko's theorem [**13**] (details can be found in [**26**]). In [**26**] we proposed a general family $F_{k,n}$ of well-poised hypergeometric series, close to Rivoal's series in [**16**] and [**17**], and conjectured that they coincide with the integrals (4). The coincidence gives a simple way to evaluate asymptotics of the integrals $J_{k,n}$ for k fixed, $n\to\infty$, and

automatically implies the properties

$$\begin{aligned} 2^{k-2}D_n^{k+1}J_{k,n} &\in \mathbb{Z}\zeta(k)+\mathbb{Z}\zeta(k-2)+\cdots+\mathbb{Z}\zeta(4)+\mathbb{Z}\zeta(2)+\mathbb{Z} \quad \text{for } k \text{ even},\\ D_n^{k+1}J_{k,n} &\in \mathbb{Z}\zeta(k)+\mathbb{Z}\zeta(k-2)+\cdots+\mathbb{Z}\zeta(5)+\mathbb{Z}\zeta(3)+\mathbb{Z} \quad \text{for } k \text{ odd}, \end{aligned} \tag{9}$$

which are, up to the extra factor D_n, inclusions (3) and (6) when $k = 2, 3, 4, 5$. Inclusions (9) give a partial answer to a conjecture of Vasilyev [**21**]. Finally, we were able [**27**], [**28**] to prove the conjectured coincidence for any integer $k \geq 2$. We refer the interested reader to the recent survey [**9**] for further information.

The important point is that both Euler-type multiple integral and hypergeometric representation of $\mathbb{Q}$-linear forms in zeta values (the values of Riemann's zeta function at positive integers) can be successfully applied for studying analytic and arithmetic properties of the forms. As an example of such use, we may mention that it covers the transformation groups from [**14**] and [**15**]; the groups are the crucial ingredient in the remarkable arithmetic approach of Rhin and Viola. Due to [**7**], [**14**], [**15**], [**21**], [**22**], it is natural to expect that dealing with Euler-type multiple integrals instead of well-poised hypergeometric objects could remove the extra factor D_n in (9) (cf. the denominator conjectures in [**17**], [**26**], [**28**]).

Here we deduce new theorems that represent well-poised hypergeometric series and integrals as Euler-type multiple integrals (see Theorems 2, 5 and 6). The theorems allow us to state new transformation formulae for the multiple integrals (Theorems 3, 4 and 7); the formulae cannot be covered by any evident change of variable. In addition, we are able to present a 5-fold integral for certain linear forms in $\mathbb{Q}\zeta(4)+\mathbb{Q}$ (see (17) below) revealed in [**28**] in terms of a hypergeometric integral.

2. *Connection theorems*

In the announcement [**27**] (with details in [**28**]), we connect the (very-)well-poised hypergeometric series

$$\begin{aligned} F_k(\boldsymbol{h}) = F_k(h_0;h_1,\dots,h_k) &= \frac{\Gamma(1+h_0)\prod_{j=1}^k\Gamma(h_j)}{\prod_{j=1}^k\Gamma(1+h_0-h_j)}\\ &\quad\times {}_{k+2}F_{k+1}\left(\begin{matrix} h_0,\, 1+\frac12 h_0, & h_1, & \dots, & h_k\\ \frac12 h_0, & 1+h_0-h_1, & \dots, & 1+h_0-h_k\end{matrix}\,\middle|\,(-1)^{k+1}\right)\\ &= \sum_{\mu=0}^{\infty}(h_0+2\mu)\frac{\prod_{j=0}^k\Gamma(h_j+\mu)}{\prod_{j=0}^k\Gamma(1+h_0-h_j+\mu)}(-1)^{(k+1)\mu} \end{aligned} \tag{10}$$

and the multiple integrals

$$\begin{aligned} J_k(\boldsymbol{a},\boldsymbol{b}) &= J_k\left(\begin{matrix} a_0,\, a_1,\, \dots,\, a_k\\ b_1,\, \dots,\, b_k\end{matrix}\right)\\ &= \int\limits_{[0,1]^k}\cdots\int \frac{\prod_{j=1}^k x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{Q_k(x_1,\dots,x_k)^{a_0}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k, \end{aligned} \tag{11}$$

where $Q_k(x_1,\dots,x_k)$ is defined in (5) for $k \geq 1$ (and by agreement $J_0(a_0) = 1$). Here all the coordinates of $\boldsymbol{h} = (h_0;h_1,\dots,h_k)$, $\boldsymbol{a} = (a_0,a_1,\dots,a_k)$, $\boldsymbol{b} = (b_1,\dots,b_k)$ are complex numbers satisfying natural conditions for convergence; namely, the lower parameters in (10) are not negative integers and

$$\operatorname{Re} b_j > \operatorname{Re} a_j > 0 \quad \text{and} \quad \operatorname{Re}(b_j-a_j) > \operatorname{Re} a_0 \qquad \text{for } j = 1,\dots,k, \tag{12}$$

while additional conditions on $\boldsymbol{h}$ read as follows:

$$\begin{gathered} 1+\operatorname{Re} h_0 > \frac{2}{k-1}\cdot\sum_{j=1}^{k}\operatorname{Re} h_j, \\ \operatorname{Re}(1+h_0-h_j-h_{j+1})>0 \quad \text{for } j=1,\dots,k-1. \end{gathered} \tag{13}$$

THEOREM 1 ([**27**], [**28**]). *For each $k\ge 2$, the identity*

$$\begin{aligned} &\frac{\prod_{j=1}^{k-1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\,\Gamma(h_k)}\cdot F_k(h_0;h_1,\dots,h_k) \\ &\quad = J_{k-2}\biggl(\begin{matrix} h_1, & h_2, & h_3, & \dots, & h_{k-1} \\ & 1+h_0-h_3, & 1+h_0-h_4, & \dots, & 1+h_0-h_k \end{matrix}\biggr) \end{aligned}$$

holds, provided that the objects on the left and on the right are well defined.

Specializing $h_0=3n+2$ and $h_1=\dots=h_k=n+1$ in Theorem 1, we obtain the integral $J_{k-2,n}$ defined in (4); therefore Rivoal's arguments in [**16**] imply inclusions (9).

Let $k\ge 2$ be even. Define the well-poised hypergeometric integrals

$$\begin{aligned} F'_k(\boldsymbol{h}) &= F'_k(h_0,h_{-1};h_1,h_2,\dots,h_k) \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(h_0+2t)\frac{\prod_{j=-1}^{k}\Gamma(h_j+t)\cdot\Gamma(h_{-1}-h_0-t)\,\Gamma(-t)}{\prod_{j=1}^{k}\Gamma(1+h_0-h_j+t)}\,\mathrm{d}t \end{aligned}$$

and the Euler-type $(k-1)$-fold integrals

$$\begin{aligned} J'_{k-1}(\boldsymbol{a},\boldsymbol{b}) &= J'_{k-1}\biggl(\begin{matrix} a_0, & a_1, & \dots, & a_{k-1} \\ & b_1, & \dots, & b_{k-1} \end{matrix}\biggr) \\ &= \int\cdots\int_{[0,1]^{k-1}}\frac{\prod_{j=1}^{k-1}x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{Q'_{k-1}(x_1,\dots,x_{k-1})^{a_0}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{k-1}, \end{aligned}$$

where $Q'_1=1$ and

$$Q'_{k+1}(x_1,\dots,x_{k+1}) = (1-x_1)(1-x_{k+1}) + Q'_{k-1}(x_1,\dots,x_{k-1})\cdot x_k x_{k+1}$$

for $k\ge 2$. (We prefer to mark the new objects by primes; this should not trouble the reader since we do not use the symbol for differentiation.)

THEOREM 2. *For even $k\ge 2$, we have*

$$\begin{aligned} &\frac{\Gamma(1+h_0-h_1-h_k)\,\prod_{j=2}^{k-1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_{-1})\,\Gamma(h_1)\,\Gamma(h_{-1}-h_0+h_2)}\cdot F'_k(h_0,h_{-1};h_1,\dots,h_k) \\ &\quad = J'_{k-1}\biggl(\begin{matrix} h_1, & h_2, & h_3, & \dots, & h_k \\ & h_{-1}-h_0+h_1+h_2, & 1+h_0-h_2, & \dots, & 1+h_0-h_{k-1} \end{matrix}\biggr), \end{aligned} \tag{14}$$

provided that the objects on the left and on the right are well defined.

The theorem yields that, under the hypothesis

$$a_1+b_2=a_2+b_3=a_3+b_4=\dots=a_{k-1}+b_k, \qquad k\ge 1 \text{ odd},$$

the multiple integral $J'_k(\boldsymbol{a},\boldsymbol{b})$ can be translated into a (1-fold, i.e., single) Barnes-type integral.

3. *A hypergeometric proof of Theorem* 2

To avoid technical complications, we would like to make a preliminary agreement. The process of interchanging the order of integration (when it is required) may be justified by standard methods similar to those used by Barnes [**6**] and Bailey [**5**] (see also [**25**], Chapter 14). The procedure usually requires restricting convergence domains for parameters; the restrictions on the parameters may be removed in the final results by analytic continuation.

Lemma 1. *Theorem* 2 *is true for* $k=2$.

Proof. If $k=2$, then

$$F'_2(h_0,h_{-1};h_1,h_2)=\frac{\Gamma(h_{-1})\,\Gamma(h_1)\,\Gamma(h_2)\,\Gamma(h_{-1}-h_0+h_1)\,\Gamma(h_{-1}-h_0+h_2)}{\Gamma(1+h_0-h_1-h_2)\,\Gamma(h_{-1}-h_0+h_1+h_2)}$$

by the integral analogue of the ${}_5F_4(1)$ summation theorem (see [**4**], Section 7). Hence

$$\begin{aligned}J'_1\left(\begin{matrix}h_1, & h_2\\ & h_{-1}-h_0+h_1+h_2\end{matrix}\right)&=\int_0^1 x_1^{h_2-1}(1-x_1)^{h_{-1}-h_0+h_1-1}\,\mathrm{d}x_1\\&=\frac{\Gamma(h_2)\,\Gamma(h_{-1}-h_0+h_1)}{\Gamma(h_{-1}-h_0+h_1+h_2)}\\&=\frac{\Gamma(1+h_0-h_1-h_2)}{\Gamma(h_{-1})\,\Gamma(h_1)\,\Gamma(h_{-1}-h_0+h_2)}\cdot F'_2(h_0,h_{-1};h_1,h_2).\end{aligned}\qquad\square$$

The following result is the particular case $m=3$, $r=2$ of Theorem 2 in [**13**], Section 3.2.

Lemma 2. *Let* a_0,a_1,a_2 *and* b_1,b_2 *be complex numbers satisfying the conditions* $\operatorname{Re}a_0>0$, $\operatorname{Re}b_j>\operatorname{Re}a_j>0$ *and* $\operatorname{Re}(b_1-a_1-a_0)>-\operatorname{Re}a_j$ *for* $j=1,2$. *Then for any* $z\in\mathbb{C}$ *with* $|\arg z|<\pi$ *the following is a valid relation between convergent integrals:*

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a_0+s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)\,\Gamma(b_1-a_1-a_0-s)\,\Gamma(-s)}{\Gamma(b_2+s)}\,z^s\,\mathrm{d}s\\&\quad=\frac{\Gamma(a_0)\,\Gamma(b_1-a_0)}{\Gamma(b_2-a_2)}\iint\limits_{[0,1]^2}\frac{u^{a_1-1}(1-u)^{b_1-a_1-1}v^{a_2-1}(1-v)^{b_2-a_2-1}}{(1-u+zuv)^{a_0}}\,\mathrm{d}u\,\mathrm{d}v,\end{aligned}$$

where $z^s=e^{s\log|z|+is\arg z}$.

Proof of Theorem 2. We begin as in [**5**], Section 6.6, by writing the following

consequence of Barnes's second lemma:

$$\frac{\Gamma(h_1+t)\,\Gamma(h_{k+1}+t)\,\Gamma(h_{k+2}+t)}{\Gamma(1+h_0-h_1+t)\,\Gamma(1+h_0-h_{k+1}+t)\,\Gamma(1+h_0-h_{k+2}+t)}$$
$$=\frac{1}{\Gamma(1+h_0-h_1-h_{k+1})\,\Gamma(1+h_0-h_1-h_{k+2})\,\Gamma(1+h_0-h_{k+1}-h_{k+2})}$$
$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\begin{array}{l}\Gamma(h_1+s)\,\Gamma(h_{k+1}+s)\,\Gamma(h_{k+2}+s)\\ \times\Gamma(1+h_0-h_1-h_{k+1}-h_{k+2}-s)\,\Gamma(t-s)\end{array}}{\Gamma(1+h_0+t+s)}\,\mathrm{d}s.$$

Assuming that (14) is true for a given even $k\ge 2$, the above formula implies

$$F'_{k+2}(h_0,h_{-1};h_1,h_2,\dots,h_k,h_{k+1},h_{k+2})$$
$$=\frac{1}{\Gamma(1+h_0-h_1-h_{k+1})\,\Gamma(1+h_0-h_1-h_{k+2})\,\Gamma(1+h_0-h_{k+1}-h_{k+2})}$$
$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(h_1+s)\,\Gamma(h_{k+1}+s)\,\Gamma(h_{k+2}+s)$$
$$\times\Gamma(1+h_0-h_1-h_{k+1}-h_{k+2}-s)\cdot F'_k(h_0,h_{-1};-s,h_2,\dots,h_k)\,\mathrm{d}s$$
$$=\frac{\Gamma(h_{-1})\,\Gamma(h_{-1}-h_0+h_2)\,\Gamma(1+h_0-h_k-h_{k+1})}{\Gamma(1+h_0-h_1-h_{k+1})\,\Gamma(1+h_0-h_1-h_{k+2})\,\prod_{j=2}^{k+1}\Gamma(1+h_0-h_j-h_{j+1})}$$
$$\times\int\limits_{[0,1]^{k-1}}\!\!\cdots\!\int x_1^{h_2-1}(1-x_1)^{h_{-1}-h_0-1}\prod_{j=2}^{k-1}x_j^{h_{j+1}-1}(1-x_j)^{h_0-h_j-h_{j+1}}$$
$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\begin{array}{l}\Gamma(h_1+s)\,\Gamma(h_{k+1}+s)\,\Gamma(h_{k+2}+s)\\ \times\Gamma(1+h_0-h_1-h_{k+1}-h_{k+2}-s)\,\Gamma(-s)\end{array}}{\Gamma(1+h_0-h_k+s)}\left(\frac{Q'_{k-1}}{1-x_1}\right)^s\mathrm{d}s$$
$$\times\,\mathrm{d}x_1\cdots\mathrm{d}x_{k-1},$$

where $Q'_{k-1}=Q'_{k-1}(x_1,\dots,x_{k-1})$. Applying Lemma 2 with $u=x_{k+1}$, $v=x_k$, $a_0=h_1$, $a_1=h_{k+2}$, $a_2=h_{k+1}$, $b_1=1+h_0-h_{k+1}$, $b_2=1+h_0-h_k$ and $z=Q'_{k-1}/(1-x_1)$, and using

$$1-x_{k+1}+\frac{Q'_{k-1}(x_1,\dots,x_{k-1})x_kx_{k+1}}{1-x_1}=\frac{Q'_{k+1}(x_1,\dots,x_{k+1})}{1-x_1},$$

we arrive at the desired identity (14) for $k+2$ instead of k. Since the inductive base $k=2$ is covered by Lemma 1, the theorem is proved. □

4. *Transformations of multiple integrals*

To derive transformations for the multiple integrals J'_k, where $k\ge 1$ is odd, we first assign a set $\boldsymbol{e}$ of $(k+1)(k+4)/2$ complementary parameters to the set $\boldsymbol{h}=(h_0,h_{-1};h_1,\dots,h_{k+1})$ by the following rule:

$$\begin{gathered}e_{jl}=h_0-h_j-h_l,\quad 1\le j<l\le k+1,\\ e_{0l}=h_l-1\ \text{ and }\ \bar e_{0l}=h_{-1}-h_0+h_l-1,\quad 1\le l\le k+1.\end{gathered}\tag{15}$$

It will be convenient to assume that the real parts of all elements in $\boldsymbol{e}$ are >-1; this condition ensures the convergence of the multiple integrals in the following

discussion. Finally, set

$$H'_k(\boldsymbol{e}) = J'_k\begin{pmatrix} h_1, & h_2, & h_3, & \dots, & h_{k+1} \\ & h_{-1}-h_0+h_1+h_2, & 1+h_0-h_2, & \dots, & 1+h_0-h_k \end{pmatrix}$$
$$= \int\limits_{[0,1]^k}\cdots\int \frac{x_1^{e_{02}}(1-x_1)^{\overline{e}_{01}}\cdot\prod_{j=2}^k x_j^{e_{0,j+1}}(1-x_j)^{e_{j,j+1}}}{Q'_k(x_1,\dots,x_k)^{e_{01}+1}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k. \tag{16}$$

Note that *any permutation of the parameters $h_1,\dots,h_{k+1}$ induces a permutation of the parameters in the set $\boldsymbol{e}$.* (It would be a hard job to find corresponding changes of variable of the multiple integral (16).) Denote by $\mathfrak{G}_k \subset \mathfrak{S}_{(k+1)(k+4)/2}$ the $\boldsymbol{e}$-permutation group generated by all permutations of the parameters $h_1,\dots,h_{k+1}$ hence $\mathfrak{G}_k \cong \mathfrak{S}_{k+1}$. Thanks to Theorem 2 we obtain that the quantity

$$\frac{H'_k(\boldsymbol{e})}{\prod_{j=2}^{k+1}\Gamma(e_{0j}+1)\cdot\Gamma(\overline{e}_{01}+1)\prod_{j=3}^{k+1}\Gamma(\overline{e}_{0j}+1)\cdot\Gamma(e_{1,k+1}+1)\prod_{j=2}^{k}\Gamma(e_{j,j+1}+1)}$$
$$= \frac{F'_{k+1}(\boldsymbol{h})}{\Gamma(h_{-1})\,\prod_{j=1}^{k+1}\Gamma(h_j)\,\prod_{j=1}^{k+1}\Gamma(h_{-1}-h_0+h_j)}$$

is stable (i.e. fixed) under the action of the group $\mathfrak{G}_k$. Put

$$\mathcal{E} = \mathcal{E}(\boldsymbol{e}) = \{e_{02}, e_{03}, \dots, e_{0,k+1},\ \overline{e}_{01}, \overline{e}_{03}, \overline{e}_{04}, \dots, \overline{e}_{0,k+1},\ e_{1,k+1}, e_{23}, e_{34}, \dots, e_{k,k+1}\}$$

and

$$\Pi(\boldsymbol{e}) = \prod_{e_{jl}\in\mathcal{E}}\Gamma(e_{jl}+1), \qquad \Sigma(\boldsymbol{e}) = \sum_{e_{jl}\in\mathcal{E}} e_{jl}.$$

In this notation, we obtain the following result.

Theorem 3 (Transformation theorem for multiple integrals). *For odd $k \geq 1$, the quantities $H'_k(\boldsymbol{e})/\Pi(\boldsymbol{e})$ and $\Sigma(\boldsymbol{e})$ are $\mathfrak{G}_k$-stable.*

The last claim follows from the computation

$$\Sigma(\boldsymbol{e}) = \sum_{\substack{j=1\\ j\neq 1}}^{k+1}(h_j-1) + \sum_{\substack{j=1\\ j\neq 2}}^{k+1}(h_{-1}-h_0+h_j-1)$$
$$+ (h_0-h_1-h_{k+1}) + \sum_{j=2}^{k}(h_0-h_j-h_{j+1}) = k(h_{-1}-2)$$

and $\mathfrak{G}_k$-stability of h_{-1}.

5. *Integrals involving* $\zeta(4)$

Let $k=6$ in Theorem 2 and let all the parameters in $\boldsymbol{h}$ be integers, so that all the parameters in $\boldsymbol{e}$ are non-negative (as we agreed) integers. We mention the inclusion $F'_6(\boldsymbol{h}) \in \mathbb{Q}\zeta(4)+\mathbb{Q}$, proved in [**28**] by considering the series for the well-poised hypergeometric integral. Thus Theorem 2 yields the inclusion $H'_5(\boldsymbol{e}) \in \mathbb{Q}\zeta(4)+\mathbb{Q}$ for the corresponding multiple integral.

One of the interesting consequences of the case

$$h_0 = h_{-1} = 3n+2, \qquad h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = n+1$$

(i.e., all elements in $\boldsymbol{e}$ equal n) is a second-order Apéry-like recursion for $\zeta(4)$ (see [**28**]; the recursion was obtained by other means in [**8**] and [**19**]). Here is another consequence of this case:

$$-\sum_{\nu=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\biggl((2t+n)\biggl(\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^2}\biggr)^2\biggr)\biggr|_{t=\nu}$$
$$=\frac{(3n+1)!}{n!^3}\int\limits_{[0,1]^5}\cdots\int\frac{\prod_{j=1}^5 x_j^n(1-x_j)^n\,\mathrm{d}x_1\cdots\mathrm{d}x_5}{Q(x_1,x_2,x_3,x_4,x_5)^{n+1}}\in\mathbb{Q}\zeta(4)+\mathbb{Q}, \tag{17}$$

where

$$\begin{aligned}Q(x_1,x_2,x_3,x_4,x_5)&=Q_5'(1-x_1,x_2,x_3,x_4,x_5)\\&=x_1(1-(1-(1-(1-x_2)x_3)x_4)x_5)+(1-x_1)x_2x_3x_4x_5.\end{aligned}$$

The integral on the right-hand side of (17) converges since it is finite for $n=0$ (when it equals $6\zeta(4)$) and

$$\max_{\substack{x_j\in[0,1]\\ j=1,\dots,5}}\frac{3^3\prod_{j=1}^5 x_j(1-x_j)}{Q(x_1,x_2,x_3,x_4,x_5)}=(2\sqrt{3}-3)^3$$

(the maximum is achieved at the point $x_1=(3-\sqrt{3})/6$, $x_2=(3-\sqrt{3})/3$, $x_3=2\sqrt{3}-3$, $x_4=(\sqrt{3}-1)/2$, $x_5=\sqrt{3}/3$). Theorem 2 and identity (17) solve our problem in [**28**] of determining a 5-fold integral for linear forms in 1 and $\zeta(4)$. In [**19**], Sorokin gives a less economic 6-fold integral for the same object; we also mention that our 'arithmetic' normalization of $H_5'(\boldsymbol{e})$ differs from that in [**28**] (the additional gamma factors appear due to the method of construction).

The exceptionality of the multiple integral $H_5'(\boldsymbol{e})$ consists in the fact that, under the hypothesis

$$2+3h_0=h_{-1}+h_1+h_2+h_3+h_4+h_5+h_6 \tag{18}$$

on the parameters $\boldsymbol{h}$, the additional transformation

$$\begin{aligned}\boldsymbol{h}\mapsto(&1+2h_0-h_1-h_2-h_3,h_{-1};1+h_0-h_2-h_3,\\&1+h_0-h_1-h_3,1+h_0-h_1-h_2,h_4,h_5,h_6)\end{aligned}$$

(with the induced $\boldsymbol{e}$-permutation denoted by $\mathfrak{b}$; see [**28**], formula (54)) is known due to Bailey [**4**], Section 9. In combination with Theorem 3, this fact yields the following result.

THEOREM 4. *Under hypothesis* (18)*, the quantities* $H_5'(\boldsymbol{e})/\Pi(\boldsymbol{e})$ *and* $\Sigma(\boldsymbol{e})$ *are* $\langle\mathfrak{b},\mathfrak{S}_5\rangle$*-stable.*

In [**28**], by means of hypergeometric integrals we use Theorem 4 to deduce a certain conditional result on the arithmetic properties of $\zeta(4)$. In [**28**] we show that the group $\langle\mathfrak{b},\mathfrak{S}_5\rangle$ is rather rich and contains $72\cdot 6!=51840$ permutations.

REMARK. We have verified that there is no way to cover Bailey's transform $\mathfrak{b}$ (or any other permutation from $\langle\mathfrak{b},\mathfrak{S}_5\rangle\setminus\mathfrak{S}_5$) by doing 'evident' changes of variable, i.e., rearranging the variables $x_1,\dots,x_5$ and/or replacing some x_j by $1-x_j$. Therefore, we suspect that there exists an ingenious change of variable in the corresponding integral J_5' (for instance, coming from a birational transformation technique [**22**]).

6. *Another connection theorem*

Consider now the general family of very-well-poised hypergeometric series

$$\begin{aligned} F_k(\boldsymbol{h} \mid z) = F_k(h_0; h_1, \dots, h_k \mid z) &= \frac{\Gamma(1+h_0)\prod_{j=1}^k \Gamma(h_j)}{\prod_{j=1}^k \Gamma(1+h_0-h_j)} \\ &\quad \times {}_{k+2}F_{k+1}\left(\begin{matrix} h_0, & 1+\frac12 h_0, & h_1, & \dots, & h_k \\ & \frac12 h_0, & 1+h_0-h_1, & \dots, & 1+h_0-h_k \end{matrix} \,\middle|\, (-1)^{k+1} z\right) \\ &= \sum_{\mu=0}^{\infty} (h_0+2\mu) \frac{\prod_{j=0}^k \Gamma(h_j+\mu)}{\prod_{j=0}^k \Gamma(1+h_0-h_j+\mu)} (-1)^{(k+1)\mu} z^\mu \end{aligned} \tag{19}$$

and the multiple integrals

$$\begin{aligned} J_k(\boldsymbol{a}, \boldsymbol{b} \mid z) &= J_k\left(\begin{matrix} a_0, & a_1, & \dots, & a_k \\ & b_1, & \dots, & b_k \end{matrix} \,\middle|\, z\right) \\ &= \int\limits_{[0,1]^k} \cdots \int \frac{\prod_{j=1}^k x_j^{a_j-1}(1-x_j)^{b_j-a_j-1}}{Q_k(x_1, \dots, x_k \mid z)^{a_0}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k \end{aligned} \tag{20}$$

with $Q_0(z) \equiv 1$ and

$$\begin{aligned} Q_k(z) = Q_k(x_1, \dots, x_k \mid z) &= Q_{k-1}(x_1, \dots, x_{k-1} \mid z) + (-1)^k z x_1 x_2 \cdots x_k \\ &= 1 - z x_1 (1 - x_2 (1 - \cdots (1 - x_{k-1}(1-x_k)) \cdots)) \qquad \text{for } k \geq 1 \end{aligned}$$

(with the agreement that $J_0(a_0 \mid z)$ is identically 1). Omitting the parameter z means specializing $z = 1$, which agrees with the notation of Section 2. As before, we require conditions (12), (13) for the complex parameters $\boldsymbol{h}$, $\boldsymbol{a}$, $\boldsymbol{b}$ to ensure the convergence of (19) and (20) in the unit disk $|z| < 1$ (and even $|z| \leq 1$).

The proof in [**28**] of Theorem 1, as well as our proof here of Theorem 2, is based on Barnes-type integral representations of the series in question. We now apply 'series' arguments in order to prove a connection between (19) and (20) for generic $z \in \mathbb{C}$.

THEOREM 5. *For each $k \geq 2$, we have the identity*

$$\begin{aligned} &\frac{\prod_{j=1}^{k-1} \Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\Gamma(h_k)} \cdot F_k(h_0; h_1, \dots, h_k \mid z) \\ &\quad = \frac{\Gamma(1+h_0)}{\Gamma(1+\frac12 h_0)\Gamma(\frac12 h_0 - h_1)\Gamma(h_1)} \cdot \frac{1-z}{(1+z)^{1+h_0}} \\ &\qquad \times J_k\left(\begin{matrix} \frac12+\frac12 h_0, & 1+\frac12 h_0, & h_1, & \dots, & h_{k-1} \\ & 1+h_0-h_1, & 1+h_0-h_2, & \dots, & 1+h_0-h_k \end{matrix} \,\middle|\, \frac{4z}{(1+z)^2}\right), \end{aligned} \tag{21}$$

provided that the objects on the left and on the right are well defined.

To ensure the convergence of both sides in (21), it is sufficient to prove the formula for the disk $|z| < 3 - 2\sqrt{2}$, which is contained in the interior of the curve $4|z| = |1+z|^2$. This restriction brings no loss of generality, since the complete result follows by analytic continuation.

Lemma 3. *Theorem 5 holds when $k=2$.*

Proof. Thanks to Bailey [**3**], we have

$$
\begin{aligned}
&{}_4F_3\biggl(\begin{matrix} a,\ 1+\frac12 a, & b, & c \\ \frac12 a, & 1+a-b, & 1+a-c \end{matrix}\biggr|\, -z\biggr) \\
&\qquad = \frac{1-z}{(1+z)^{a+1}}\cdot {}_3F_2\biggl(\begin{matrix} \frac12+\frac12 a, & 1+\frac12 a, & 1+a-b-c \\ & 1+a-b, & 1+a-c \end{matrix}\biggr|\, \frac{4z}{(1+z)^2}\biggr).
\end{aligned}
\tag{22}
$$

Using the Euler-type integral representation

$$
\begin{aligned}
{}_3F_2\biggl(\begin{matrix} a_0,\ a_1,\ a_2 \\ b_1,\ b_2 \end{matrix}\biggr|\, z\biggr) &= \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(b_1-a_1)\,\Gamma(b_2-a_2)} \\
&\qquad \times \iint\limits_{[0,1]^2} \frac{u^{a_1-1}(1-u)^{b_1-a_1-1}v^{a_2-1}(1-v)^{b_2-a_2-1}}{(1-zuv)^{a_0}}\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

for the right-hand side of (22), and the change of variable $u=x_1$, $v=1-x_2$, we deduce the desired result. □

Lemma 4. *Let z be a complex number satisfying $|z|<\frac12$. Then, for each integer $k\ge 2$, we have the relation*

$$
\begin{aligned}
J_k\biggl(\begin{matrix} a_0,\ a_1,\ \dots,\ a_{k-1},\ a_k \\ b_1,\ \dots,\ b_{k-1},\ b_k \end{matrix}\biggr|\, z\biggr) &= \frac{\Gamma(b_k-a_k)}{\Gamma(a_0)}\sum_{\nu=0}^{\infty}\frac{\Gamma(a_0+\nu)\,\Gamma(a_k+\nu)}{\Gamma(1+\nu)\,\Gamma(b_k+\nu)} \\
&\qquad \times J_{k-1}\biggl(\begin{matrix} a_0+\nu,\ a_1+\nu,\ \dots,\ a_{k-1}+\nu \\ b_1+\nu,\ \dots,\ b_{k-1}+\nu \end{matrix}\biggr|\, z\biggr)(-1)^{(k-1)\nu}z^{\nu}.
\end{aligned}
$$

Proof. Writing

$$
Q_k(z) = Q_{k-1}(z) - (-1)^{k-1}zx_1\cdots x_{k-1}x_k = Q_{k-1}(z)\cdot(1-Zx_k),
$$

where

$$
Z = \frac{(-1)^{k-1}zx_1\cdots x_{k-1}}{Q_{k-1}(x_1,\dots,x_{k-1}\mid z)}
$$

satisfies

$$
|Z| = \frac{|z|\cdot x_1\cdots x_{k-1}}{|1-z(x_1-x_1x_2+x_1x_2x_3-\dots+(-1)^k x_1\cdots x_{k-1})|} \le \frac{|z|}{1-|z|} < 1,
$$

and using the formula

$$
\begin{aligned}
\int_0^1 \frac{x_k^{a_k-1}(1-x_k)^{b_k-a_k-1}}{(1-Zx_k)^{a_0}}\,\mathrm{d}x_k &= \frac{\Gamma(a_k)\,\Gamma(b_k-a_k)}{\Gamma(b_k)}\cdot {}_2F_1\biggl(\begin{matrix} a_0,\ a_k \\ b_k \end{matrix}\biggr|\, Z\biggr) \\
&= \frac{\Gamma(b_k-a_k)}{\Gamma(a_0)}\sum_{\nu=0}^{\infty}\frac{\Gamma(a_0+\nu)\,\Gamma(a_k+\nu)}{\nu!\,\Gamma(b_k+\nu)}Z^{\nu},
\end{aligned}
$$

we arrive at the desired relation. □

Remark. Since

$$
\frac{4|z|}{|1+z|^2} \le \frac{4|z|}{(1-|z|)^2} < \frac{4(3-2\sqrt2)}{(2\sqrt2-2)^2} = \sqrt2-1 < \frac12
$$

for $|z|<3-2\sqrt2$, we may apply Lemma 4 substituting $4z/(1+z)^2$ for z.

Proof of Theorem 5. The case $k = 2$ is covered by Lemma 3. We assume that $k \geq 2$ and that identity (21) holds for k, hence

$$\begin{aligned}
&\frac{\Gamma(1+h_0+2\nu)}{\Gamma(1+\frac{1}{2}h_0+\nu)\,\Gamma(\frac{1}{2}h_0-h_1)\,\Gamma(h_1+\nu)}\cdot\frac{1-z}{(1+z)^{1+h_0+2\nu}}\\
&\quad\times J_k\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}h_0+\nu,\ 1+\frac{1}{2}h_0+\nu,\ h_1+\nu,\ \dots,\ h_{k-1}+\nu\\ 1+h_0-h_1+\nu,\ 1+h_0-h_2+\nu,\ \dots,\ 1+h_0-h_k+\nu\end{matrix}\,\middle|\,\frac{4z}{(1+z)^2}\right)\\
&\qquad=\frac{\prod_{j=1}^{k-1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1+\nu)\,\Gamma(h_k+\nu)}\cdot F_k(h_0+2\nu;h_1+\nu,\dots,h_k+\nu\mid z)
\end{aligned}\tag{23}$$

for $\nu = 0, 1, 2, \dots$. Then applying the inductive hypothesis (23), Lemma 4 (with $4z/(1+z)^2$ in place of z), and the duplication formula for the gamma function, and reordering double summation, we conclude that

$$\begin{aligned}
&\frac{\Gamma(1+h_0)}{\Gamma(1+\frac{1}{2}h_0)\,\Gamma(\frac{1}{2}h_0-h_1)\,\Gamma(h_1)}\cdot\frac{1-z}{(1+z)^{1+h_0}}\\
&\quad\times J_{k+1}\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}h_0, & 1+\frac{1}{2}h_0, & h_1, & \dots, & h_k\\ & 1+h_0-h_1, & 1+h_0-h_2, & \dots, & 1+h_0-h_{k+1}\end{matrix}\,\middle|\,\frac{4z}{(1+z)^2}\right)\\
&\qquad=\frac{\Gamma(1+h_0)}{\Gamma(1+\frac{1}{2}h_0)\,\Gamma(\frac{1}{2}h_0-h_1)\,\Gamma(h_1)}\cdot\frac{1-z}{(1+z)^{1+h_0}}\cdot\frac{\Gamma(1+h_0-h_k-h_{k+1})}{\Gamma(\frac{1}{2}+\frac{1}{2}h_0)}\\
&\qquad\quad\times\sum_{\nu=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}h_0+\nu)\,\Gamma(h_k+\nu)}{\Gamma(1+\nu)\,\Gamma(1+h_0-h_{k+1}+\nu)}\left((-1)^k\frac{4z}{(1+z)^2}\right)^{\nu}\\
&\qquad\quad\times J_k\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}h_0+\nu,\ 1+\frac{1}{2}h_0+\nu,\ h_1+\nu,\ \dots,\ h_{k-1}+\nu\\ 1+h_0-h_1+\nu,\ 1+h_0-h_2+\nu,\ \dots,\ 1+h_0-h_k+\nu\end{matrix}\,\middle|\,\frac{4z}{(1+z)^2}\right)\\
&\qquad=\frac{\Gamma(1+h_0)\,\prod_{j=1}^{k}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(\frac{1}{2}+\frac{1}{2}h_0)\,\Gamma(1+\frac{1}{2}h_0)\,\Gamma(h_1)}\\
&\qquad\quad\times\sum_{\nu=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}h_0+\nu)\,\Gamma(1+\frac{1}{2}h_0+\nu)}{\Gamma(1+h_0+2\nu)\,\Gamma(1+\nu)\,\Gamma(1+h_0-h_{k+1}+\nu)}(-1)^{k\nu}4^{\nu}z^{\nu}\\
&\qquad\quad\times\sum_{\mu=0}^{\infty}(h_0+2\nu+2\mu)\frac{\Gamma(h_0+2\nu+\mu)\,\prod_{j=1}^{k}\Gamma(h_j+\nu+\mu)}{\Gamma(1+\mu)\,\prod_{j=1}^{k}\Gamma(1+h_0-h_j+\nu+\mu)}(-1)^{(k+1)\mu}z^{\mu}\\
&\qquad=\frac{\prod_{j=1}^{k}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)}\sum_{n=0}^{\infty}\frac{(h_0+2n)\,\prod_{j=1}^{k}\Gamma(h_j+n)}{\prod_{j=1}^{k}\Gamma(1+h_0-h_j+n)}(-1)^{(k+1)n}z^{n}\\
&\qquad\quad\times\sum_{\nu=0}^{n}\frac{\Gamma(h_0+n+\nu)}{\Gamma(1+\nu)\,\Gamma(1+n-\nu)\,\Gamma(1+h_0-h_{k+1}+\nu)}(-1)^{\nu}\\
&\qquad=\frac{\prod_{j=1}^{k}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)}\sum_{n=0}^{\infty}\frac{(h_0+2n)\,\prod_{j=1}^{k}\Gamma(h_j+n)}{\prod_{j=1}^{k}\Gamma(1+h_0-h_j+n)}(-1)^{(k+1)n}z^{n}\\
&\qquad\quad\times\frac{\Gamma(h_0+n)}{n!\,\Gamma(1+h_0-h_{k+1})}\cdot{}_2F_1\left(\begin{matrix}-n,\ h_0+n\\ 1+h_0-h_{k+1}\end{matrix}\,\middle|\,1\right).
\end{aligned}$$

Summing the last ${}_2F_1$-series by Vandermonde's theorem

$$
{}_2F_1\left(\begin{matrix}-n,\, b\\ c\end{matrix}\,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n} = (-1)^n \frac{(1+b-c-n)_n}{(c)_n}
= (-1)^n \frac{\Gamma(1+b-c)\,\Gamma(c)}{\Gamma(1+b-c-n)\,\Gamma(c+n)}
$$

(see [**5**], the final formula in Section 1.3), we derive the desired equality (21) with $k+1$ instead of k. This completes the proof of Theorem 5. □

By (20) we may write identity (21) in the form

$$
\begin{aligned}
&\frac{\prod_{j=1}^{k-1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\,\Gamma(h_k)}\cdot F_k(h_0;h_1,\dots,h_k\mid z)\\
&\quad= \frac{\Gamma(1+h_0)}{\Gamma(1+\frac12 h_0)\,\Gamma(\frac12 h_0-h_1)\,\Gamma(h_1)}(1-z)\\
&\quad\times\int\cdots\int_{[0,1]^k}\frac{x_1^{h_0/2}(1-x_1)^{h_0/2-h_1-1}\prod_{j=2}^k x_j^{h_{j-1}-1}(1-x_j)^{h_0-h_{j-1}-h_j}}{\bigl((1+z)^2-4zx_1Q_{k-1}(x_2,\dots,x_k\mid 1)\bigr)^{(1+h_0)/2}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k;
\end{aligned}
$$

then letting z equal -1 and integrating over $[0,1]$ with respect to the variable x_1 gives the following result.

Theorem 6. *For each $k\ge 2$, we have the identity*

$$
\begin{aligned}
&\frac{\prod_{j=1}^{k-1}\Gamma(1+h_0-h_j-h_{j+1})}{\Gamma(h_1)\,\Gamma(h_k)}\cdot F_k(h_0;h_1,\dots,h_k\mid -1)\\
&\quad= \frac{\Gamma(\frac12+\frac12 h_0)}{\Gamma(\frac12+\frac12 h_0-h_1)\,\Gamma(h_1)}\cdot J_{k-1}\left(\begin{matrix}\frac12+\frac12 h_0, & h_1, & \dots, & h_{k-1}\\ & 1+h_0-h_2, & \dots, & 1+h_0-h_k\end{matrix}\right),
\end{aligned}
$$

provided that the objects on the left and on the right are well defined.

Another consequence of Theorem 5 is the fact that, under the hypothesis

$$
2a_0 = 2a_1+1 = b_1+a_2 = b_2+a_3 = \dots = b_{k-1}+a_k,
$$

the multiple integral (20) can be evaluated as a generalized hypergeometric series. In the notation

$$
\begin{gathered}
e_{jl} = h_0-h_j-h_l,\quad 1\le j<l\le k,\\
e_{00}=\tfrac12 h_0,\qquad e_{0l}=h_l-1 \ \text{ and } \ \overline{e}_{0l}=\tfrac12 h_0-h_l-1,\quad 1\le l\le k,
\end{gathered}
$$

consider the function

$$
\begin{aligned}
H_k(\boldsymbol{e}\mid z) &= J_k\left(\begin{matrix}\frac12+\frac12 h_0, & 1+\frac12 h_0, & h_1, & \dots, & h_{k-1}\\ & 1+h_0-h_1, & 1+h_0-h_2, & \dots, & 1+h_0-h_k\end{matrix}\,\middle|\, z\right)\\
&= \int\cdots\int_{[0,1]^k}\frac{x_1^{e_{00}}(1-x_1)^{\overline{e}_{01}}\prod_{j=2}^k x_j^{e_{0,j-1}}(1-x_j)^{e_{j-1,j}}}{Q_k(x_1,\dots,x_k\mid z)^{e_{00}+1/2}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k
\end{aligned}
$$

and denote by $\mathfrak{H}_k\subset\mathfrak{S}_{(k+1)(k+2)/2}$ the $\boldsymbol{e}$-permutation group generated by all permutations of the parameters $h_1,\dots,h_k$, so that $\mathfrak{H}_k\simeq\mathfrak{S}_k$. Finally, set

$$
\mathcal{E}=\mathcal{E}(\boldsymbol{e}) = \{e_{01},e_{02},\dots,e_{0,k-1},\ \overline{e}_{01},\ e_{12},e_{23},\dots,e_{k-1,k}\}
$$

and

$$\Pi(\boldsymbol{e}) = \prod_{e_{jl}\in\mathcal{E}} \Gamma(e_{jl}+1), \qquad \Sigma(\boldsymbol{e}) = \sum_{e_{jl}\in\mathcal{E}} e_{jl}.$$

Then Theorem 5 implies the following stability result.

THEOREM 7 (Transformation theorem). *The quantities* $H_k(\boldsymbol{e} \mid z)/\Pi(\boldsymbol{e})$ *and* $\Sigma(\boldsymbol{e})$ *are* $\mathfrak{H}_k$*-stable.*

7. *Some open questions*

Theorem 6 is a particular case of Theorem 1 when $h_1 = \frac{1}{2} + \frac{1}{2}h_0$. The case $h_1 = \frac{1}{2}h_0$ of Theorem 1 may be established by proving a result similar to Theorem 6 based on Whipple's transformation [**24**]

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix}\,\middle|\, -z\right) \\ &\quad = \frac{1}{(1+z)^a}\cdot {}_3F_2\left(\begin{matrix} \frac{1}{2}a, & \frac{1}{2}+\frac{1}{2}a, & 1+a-b-c \\ & 1+a-b, & 1+a-c \end{matrix}\,\middle|\, \frac{4z}{(1+z)^2}\right), \end{aligned}$$

instead of Bailey's (22). Other Euler-type multiple integrals for the series (19) are obtained in [**18**].

PROBLEM 1. Is it possible to deduce Theorem 1 from an identity generic in z (like identity (21) of Theorem 5)?

Many hypergeometric identities have appeared recently in the arithmetic study of the zeta values and multiple zeta generalizations. These identities should be special cases of some meta-identities, most of which are still unknown. As an example, we can state the following

PROBLEM 2. What is the (most) general pattern for the convergence acceleration formula

$$\begin{aligned} {}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ 2, 2, 2 \end{matrix}\,\middle|\, 1\right) &= \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} \\ &= \frac{5}{4}\cdot {}_4F_3\left(\begin{matrix} 1, 1, 1, 1 \\ \frac{3}{2}, 2, 2 \end{matrix}\,\middle|\, -\frac{1}{4}\right), \end{aligned}$$

which is an important part of Apéry's original proof [**2**]?

Of course, the most intriguing question (still open) is the irrationality of $\zeta(s)$ at $s = 5, 7, 9, \ldots$.

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Moscow Lomonosov State University
Department of Mech. and Math.
Vorobiovy Gory, GSP-2
119992 Moscow, RUSSIA

wadim@ips.ras.ru